

# Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type

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## Abstract

We construct a concise gauge invariant formulation for massless, partially massless, and massive bosonic AdS fields of arbitrary symmetry type at the level of equations of motion. Our formulation admits two equivalent descriptions: in terms of the ambient space and in terms of an appropriate vector bundle, as an explicitly local first-order BRST formalism. The second version is a parent-like formulation that can be used to generate various other formulations via equivalent reductions. In particular, we demonstrate a relation to the unfolded description of massless and partially massless fields.

## 1 Introduction

Arbitrary AdS fields can be divided into three classes according to particular values of their vacuum energy  $E_0$ . These are massive fields, massless fields and partially-massless fields that carry intermediate number of degrees of freedom. In the simplest case of totally symmetric fields the above three classes were described in [1, 2, 3, 4, 5, 6]. Starting from five dimensions the totally symmetric fields are only a special case of mixed symmetry ones.

Various approaches to mixed symmetry fields are known by now. Particularly relevant for us is the manifestly AdS invariant formulation [7] in terms of AdS tensors, which can be seen as a generalization of the Fronsdal approach [1] to totally symmetric AdS fields. Another related development has to do with the frame-like description operating with  $o(d-1, 2)$ -valued  $p$ -forms as fundamental fields [8, 9, 10, 11, 12, 13]. There are other interesting approaches to mixed-symmetry AdS fields [14, 15, 16, 17, 18, 19, 20]. Studying massive mixed symmetry AdS fields is mainly motivated by the presence of such fields in the spectrum of strings on AdS [21] (see also [22, 23] for the string-inspired approach to AdS fields).

In this paper we take a different route and extend our previous results on unitary massless mixed-symmetry fields [24] to the general case including non-unitary massless, partially massless, and massive fields.<sup>1</sup> Besides the well-known developments in the unfolded formulation of higher-spin dynamics [8, 9, 26] (for a review see [27]), the approach of [24] and the present paper has its roots in the so-called parent formulation of [28, 29, 30] and the description [31] of the Minkowski space mixed symmetry fields.

Our ideology is to keep invariance with respect to AdS algebra manifest from the very beginning. This is achieved by defining fields on the ambient space and employing the AdS invariant gauge equivalence relation. It turns out that from the ambient perspective it is more natural to use as a parameter weight  $w$  determining the radial behavior of a field instead of the energy  $E_0$  which is in fact linearly related to  $w$ . In this respect, our approach is analogous to the recently proposed description [32] of totally symmetric fields.<sup>2</sup> In terms of weight  $w$  (partially) massless fields correspond to special integer values of  $w$  while massive fields correspond to generic values of  $w$ . In the later case the gauge invariance becomes purely algebraic and can be completely eliminated.

Although the ambient space formulation is very compact and algebraically transparent its locality is not manifest. The explicitly local formulation is constructed in the next step by, roughly speaking, putting the ambient space to the fiber of a bundle over the genuine AdS space. This step is identical to the one performed in [28, 29, 31, 34] and from the first quantized point of view amounts to the Fedosov-type extension [35] (see also [36, 37, 28] for the generalizations and applications relevant in the present context) of the starting point system on the ambient space.

The algebraic structure of the proposed formulation is essentially determined by  $o(d-1, 2) - sp(2n)$  Howe dual pair [38] of  $AdS_d$  spacetime algebra  $o(d-1, 2)$  and symplectic algebra  $sp(2n)$  realized on the fiber. In particular, BRST operator of  $\Omega = \nabla + Q_p$  is a sum of  $o(d-1, 2)$  background covariant derivative  $\nabla$  built from  $o(d-1, 2)$  generators and purely algebraic part  $Q_p$  built from  $sp(2n)$  generators while the off-shell constraints are also expressed through the  $sp(2n)$  generators. Besides the value of  $w$  the difference between massive, massless, and partially-massless fields is in the form of the special off-shell constraint: for massive fields it is not present while for (partially)-massless fields it is a  $t$ -th power of the respective  $sp(2n)$  generator, where  $t$  is the “depth” of the partially massless gauge transformation [5, 39].

Both the ambient space formulation of AdS dynamics and its parent-like extension are given at the level of equations of motion only. The respective Lagrangian formulation

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<sup>1</sup>Unitary fields in AdS space are described by infinite-dimensional  $o(d-1, 2)$  UIRs with lowest energies saturating the unitarity bound  $E_0 \geq E_0(s_p, d) = s_p - p + d - 2$ , where  $s_p$  denotes the  $p$ -th spin weight [25], [1],[3]. They are called unitary massless and unitary massive fields. Fields with energies below the boundary value  $E_0(s_p, d)$  are called non-unitary and include non-unitary massless, non-unitary massive and all partially massless fields.

<sup>2</sup>It is worth mentioning that weight  $w$  can be also identified to the weight of tractors involved in the description in terms of tractor bundles. For further details see [33] and references therein.

is not constructed yet. While for particular classes of AdS fields Lagrangian covariant formulation is known in one or another form [40, 41, 42, 11, 18, 19] this is not the case for general AdS fields. We hope that the algebraic and geometric structures identified in the present paper will be also helpful in constructing Lagrangians in the general case. It is expected that the ambient space BRST Lagrangian analogous to the flat space one of [31] (see also [43, 23]) determines Lagrangian for AdS fields through a version of the radial reduction of [44, 45, 46].

The paper is structured as follows. In the next section we recall some basic algebraic facts on Howe dual realizations of  $o(d-1, 2)$  and  $sp(2n)$  algebras. Besides the standard realization of the dual algebras we need the so-called twisted realization. Then in Section 3 we describe AdS fields as ambient space tensor fields subjected to the appropriate constraints, equations of motion, and gauge transformations. We then explicitly describe the choice of the parameter  $w$  and the extra constraints that give one or another irreducible system. In Section 4 the ambient space formulation is lifted to the manifestly local formulation where the ambient space is promoted to a fiber of the appropriate vector bundle over the AdS space. Cohomology of the fiber part  $Q_p$  of the BRST operator is analyzed in Section 5 where it is shown nonvanishing in the minimal and the maximal ghost numbers only. The former is identified with the gauge module of the respective unfolded formulation while the later with the Weyl module. Massive fields are discussed in Section 6. The summary of the obtained results is given in Section 7. Appendices contain various technical details needed in the main text.

## 2 Algebraic preliminaries

### 2.1 Howe dual realizations

A usual way [1, 3, 7] to describe fields on AdS space in such a way that the isometry algebra is realized linearly is to work with tensors of AdS algebra instead of Lorentz tensors. Moreover, it is also useful to identify the space-time itself as a hyperboloid embedded in the flat ambient space so that the isometries are ambient pseudo-orthogonal transformations. Following [24] we now recall algebraic tools necessary to handle arbitrary fields on AdS space in a unified way.

Let  $X^A$ ,  $A = 0, \dots, d$  be Cartesian coordinates on the  $d+1$ -dimensional ambient space  $\mathbb{R}^{d-1,2}$ . We use the usual identification of AdS space as a hyperboloid

$$\eta_{AB}X^AX^B + 1 = 0, \quad \eta_{AB} = \text{diag}(- + \dots + -). \quad (2.1)$$

Infinitesimal isometries of the hyperboloid form a pseudo-orthogonal algebra  $o(d-1, 2)$ .

Let  $A_I^A$ , where  $A = 0, \dots, d$  and  $I = 0, \dots, n-1$  be commuting variables transforming as vectors of  $o(d-1, 2)$ . The space of functions in  $A_I^A$  is naturally an  $o(d-1, 2) - sp(2n)$ -

bimodule. More precisely,  $o(d-1, 2)$  is realized by

$$J^{AB} = A_I^A \frac{\partial}{\partial A_{BI}} - A_I^B \frac{\partial}{\partial A_{AI}} . \quad (2.2)$$

The realization of  $sp(2n)$  reads

$$T_{IJ} = A_I^A A_{JA}, \quad T_I^J = \frac{1}{2} \left\{ A_I^A, \frac{\partial}{\partial A_J^A} \right\}, \quad T^{IJ} = \frac{\partial}{\partial A_I^A} \frac{\partial}{\partial A_{JA}} . \quad (2.3)$$

These two algebras form a Howe dual pair  $o(d-1, 2) - sp(2n)$  [38]. In particular, they commute in this representation. The diagonal elements  $T_I^I$  form a basis in the Cartan subalgebra while  $T^{IJ}$  and  $T_I^J$ ,  $I < J$  are the basis elements of the appropriately chosen upper-triangular subalgebra. Let us note that  $gl(n)$  algebra is realized by the generators  $T_I^J$  as a subalgebra of  $sp(2n)$  while its  $sl(n)$  subalgebra is generated by  $T_I^J$  with  $I \neq J$ .

In what follows we also need to pick a distinguished direction in the space of oscillators  $A_I^A$ . Without loss of generality we take it along  $A_0^A$  so that from now on we consider variables  $A_0^A$  and  $A_i^A$ ,  $i = 1, \dots, n-1$  separately. In particular, we identify  $sp(2n-2) \subset sp(2n)$  subalgebra preserving the direction. We use the following notation for some of  $sp(2n-2)$  generators

$$N_i^j \equiv T_i^j = A_i^A \frac{\partial}{\partial A_j^A} \quad i \neq j, \quad N_i = N_i^i \equiv T_i^i = \frac{d+1}{2} = A_i^A \frac{\partial}{\partial A_i^A}, \quad (2.4)$$

which form  $gl(n-1)$  subalgebra, and

$$T_{ij} = A_i^A A_{jA}, \quad T^{ij} = \frac{\partial}{\partial A_i^A} \frac{\partial}{\partial A_{jA}}, \quad (2.5)$$

that complete the above set of elements to  $sp(2n-2)$  algebra.

In what follows we use two different realizations of  $sp(2n)$  generators involving  $A_0^A$  and/or  $\partial/\partial A_0^A$ .

### 2.1.1 Realization on ambient space functions

In this case we take the space of polynomials in  $A_i^A$  with coefficients in smooth functions on  $\mathbb{R}^{d+1}$  with the origin excluded. If  $X^A$  are coordinates on  $\mathbb{R}^{d+1}$  the representation for  $A_0$  and  $\frac{\partial}{\partial A_0}$  is given by

$$A_0^A = X^A, \quad \frac{\partial}{\partial A_0^A} = \frac{\partial}{\partial X^A}, \quad (2.6)$$

while the remaining variables  $A_i^A$  are represented as before.

We keep the previous notation (2.4), (2.5) for generators that do not involve  $X^A$  and/or  $\partial/\partial X^A$  while those that do are denoted by

$$\begin{aligned} \mathcal{S}_i^\dagger &= A_i^A \frac{\partial}{\partial X^A}, & \bar{\mathcal{S}}^i &= X^A \frac{\partial}{\partial A_i^A}, \\ \mathcal{S}^i &= \frac{\partial}{\partial A_i^A} \frac{\partial}{\partial X_A}, & \square_X &= \frac{\partial}{\partial X^A} \frac{\partial}{\partial X_A}. \end{aligned} \quad (2.7)$$

### 2.1.2 Twisted realization

Another possibility is to realize the dual algebras on the space of polynomials in  $A_i^A$  with coefficients in formal power series in variables  $Y^A$  such that  $A_0$  and  $\frac{\partial}{\partial A_0}$  are realized as

$$A_0^A = Y^A + V^A, \quad \frac{\partial}{\partial A_0^A} = \frac{\partial}{\partial Y^A}, \quad (2.8)$$

where  $V^A$  is some  $o(d-1, 2)$  vector normalized as  $V^A V_A = -1$ . The  $sp(2n)$  generators involving  $A_0$  are then realized by (inhomogeneous) formal differential operators on the space of “functions” in  $A_i^A$  and  $Y^A$ . We use the following notation

$$\begin{aligned} S_i^\dagger &= A_i^A \frac{\partial}{\partial Y^A}, & \bar{S}^i &= (Y^A + V^A) \frac{\partial}{\partial A_i^A}, \\ S^i &= \frac{\partial}{\partial A_i^A} \frac{\partial}{\partial Y_A}, & \square_Y &= \frac{\partial}{\partial Y^A} \frac{\partial}{\partial Y_A}. \end{aligned} \quad (2.9)$$

This realization of the dual orthogonal and symplectic algebras is referred to as twisted Howe dual realization.

The twisted realization is the same as in [31] but with  $Y^A$  replaced by  $Y^A + V^A$ . Shifting by  $V^A$  is crucial because this realization is inequivalent with the usual one (*i.e.*, the one with  $V^A = 0$ ). This happens because the change of variables  $Y^A \rightarrow Y^A + V^A$  is ill-defined in the space of formal power series. In contrast to the usual realization where highest (lowest) weight conditions of  $sp(2n-2)$  determine finite-dimensional irreducible  $o(d-1, 2)$ -modules, the inhomogeneous counterpart of these conditions can determine both finite-dimensional irreducible or infinite-dimensional  $o(d-1, 2)$ -modules. In particular, it allows one to describe finite-dimensional gauge modules and infinite-dimensional Weyl modules<sup>3</sup> associated to AdS gauge fields at the equal footing. Note that the above realization for  $n = 1, 2$  has been originally described in [29] and in [24] for general  $n$ . Analogous representation has been also used in [34] to describe conformal fields.

## 3 Ambient space description of AdS gauge fields

### 3.1 Constraints and gauge symmetries

Using realization 2.1.1 in terms of functions on  $\mathbb{R}^{d+1}/\{0\}$  with values in polynomials in  $A_i^A$  unitary massless fields on AdS can be formulated in manifestly  $o(d-1, 2)$  invariant terms [1, 3].<sup>4</sup> More precisely, the space of field configurations can be described [24] by imposing a certain parabolic subalgebra of  $sp(2n)$  followed by taking a quotient with respect to gauge transformations generated by  $S_\alpha^\dagger$  with  $\alpha = 1, \dots, p$ .

<sup>3</sup>In this case it reduces to the so-called twisted-adjoint module of [9, 26, 10, 12, 47].

<sup>4</sup>See ref. [48] for a nice review of ambient space formulation of AdS tensor calculus.

Now we extend this description to the case of not necessarily unitary and massless fields. It turns out, however, that in general one is forced to allow for higher powers of certain  $sp(2n)$  generators. Constraints to be imposed on the ambient space field  $\phi = \phi(X, A)$  are grouped as follows

**General off-shell constraints.** These are tracelessness, Young symmetry and spin weight conditions

$$T^{ij}\phi = 0, \quad N_i^j\phi = 0 \quad i < j, \quad N_i\phi = s_i\phi. \quad (3.1)$$

It follows that spin numbers are ordered as  $s_1 \geq s_2 \geq \dots \geq s_{n-1}$ . To describe generic mixed-symmetry fields it is sufficient to choose parameter  $n$  satisfying  $n \leq [\frac{d+1}{2}]$ . In odd dimensions there are also self-dual fields singled out by additional constraints involving Levi-Civita tensor but we do not consider them here.<sup>5</sup>

**Radial constraint.** The radial dependence is fixed by

$$h\phi = 0, \quad h = N_X - w, \quad N_X = X^A \frac{\partial}{\partial X^A}, \quad (3.2)$$

where  $w$  is a real number which serves as a parameter of the theory. This allows to uniquely represent a field defined on the hyperboloid in terms of the ambient space field satisfying (3.2). More explicitly, taking a new coordinate system  $(r, x^m)$  in  $\mathbb{R}^{d+1}$ , such that  $r = \sqrt{-X^2}$  is a radius and  $x^m$  are dilation-invariant coordinates  $N_X x^m = 0$ , one finds  $\phi = \phi_0(x, A) r^w$ .

**Equations of motion.** Conditions involving  $X^A$ -derivatives along the hyperboloid are to be regarded as equations of motion rather than constraints. These are given by

$$\square_X \phi = 0, \quad \mathcal{S}^i \phi = 0. \quad (3.3)$$

The last equation may be regarded as a  $\frac{\partial}{\partial X^A}$ -transversality condition.

One then postulates a gauge invariance.

**Gauge invariance.** Let us fix integer parameter  $p \leq n - 1$  and let  $\chi^\alpha = \chi^\alpha(X, A)$  for  $\alpha = 1, \dots, p$  denote gauge parameter satisfying the gauge parameter version of the above constraints. These are (3.1), (3.2), and (3.3) where the constraints involving  $N_i^j$ ,  $N_i$  and  $N_X$  are modified as

$$\begin{aligned} N_i^j \chi^\alpha + \delta_i^\alpha \delta_\beta^j \chi^\beta &= 0 \quad i < j, \\ N_i \chi^\alpha + \delta_i^\alpha \chi^\alpha - s_i \chi^\alpha &= 0, \\ (N_X - w - 1) \chi^\alpha &= 0. \end{aligned} \quad (3.4)$$

A gauge equivalence is defined by

$$\phi \sim \phi + \mathcal{S}_\alpha^\dagger \chi^\alpha. \quad (3.5)$$

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<sup>5</sup>For explicit treatment of  $AdS_5$  self-dual fields we refer to Ref. [15].

or, equivalently, the gauge transformation reads as  $\delta_\chi \phi = \mathcal{S}_\alpha^\dagger \chi^\alpha$ . One can directly check that this equivalence relation is compatible with the constraints on the field and the gauge parameter. In fact, the consistency is guaranteed because  $\mathcal{S}_\alpha^\dagger$  and the remaining constraints are generators of a subalgebra from  $sp(2n)$ .

**Tangent constraints.** In addition, fields are required not to depend on the transversal to the hyperboloid components of  $A_i$  for such values of  $i$  that the gauge invariance is preserved. This is achieved by imposing the following constraints:

$$\bar{\mathcal{S}}^{\hat{\alpha}} \phi = 0, \quad \hat{\alpha} = p + 1, \dots, n - 1. \quad (3.6)$$

**Extra constraint.** Depending on a particular value taken by parameter  $w$  the above system can be either irreducible or reducible. As we are going to see the former happens for  $w$  generic while the later corresponds to special values

$$w = s_p - p - t. \quad (3.7)$$

Here parameter  $t$  takes values  $t = 1, 2, \dots, t_{\max}$ , where  $t_{\max} = s_p - s_{p+1}$ . In this case the extra irreducibility conditions

$$(\bar{\mathcal{S}}^p)^t \phi = 0, \quad (3.8)$$

are to be imposed.

It is a matter of a direct computation that for such  $w$  constraints (3.8) and (3.2) are compatible with the gauge invariance (3.5). Compatibility of (3.8) with the remaining constraints can be directly checked using the following commutation relations of  $sl(n) \subset sp(2n)$

$$[N_j^k, \mathcal{S}_i^\dagger] = \delta_i^k \mathcal{S}_j^\dagger, \quad [\bar{\mathcal{S}}^i, N_j^k] = -\delta_j^i \bar{\mathcal{S}}^k. \quad (3.9)$$

It follows  $N_j^k$  with  $j < k$  decrease a value of index  $i$  for  $\mathcal{S}_i^\dagger$  towards its minimal value  $i = 1$  and increase it for  $\bar{\mathcal{S}}^i$  towards its maximal value  $i = n - 1$ . The subalgebra formed by constraints and gauge generators may therefore involve Young symmetrizers  $N_i^j$  along with both  $\mathcal{S}_\alpha^\dagger$  and  $\bar{\mathcal{S}}^{\hat{\alpha}}$  where  $\alpha$  necessarily starts with  $\alpha = 1$  and  $\hat{\alpha}$  necessarily ends up with  $\hat{\alpha} = n - 1$ .

Furthermore, setting  $t = 1$  and  $s \equiv s_1 = s_2 = \dots = s_p$  amounts to describing unitary gauge fields. To relate the present discussion to [24] let us note that constraints (3.6) and (3.8) are equivalent to imposing  $\bar{\mathcal{S}}^i \phi = 0$ , where index  $i$  runs all admissible values,  $i = 1, \dots, n - 1$ . This happens because in this case the respective Young tableaux have the uppermost rectangular block of the length  $s$  and height  $p$  so that fields automatically satisfy  $N_\alpha^\beta = 0$  for any  $\alpha \neq \beta$  and hence  $\bar{\mathcal{S}}^p \phi = 0$  implies  $\bar{\mathcal{S}}^\alpha \phi = 0$ .

## 3.2 Ghost variables and BRST operator

The constraints for both field and gauge parameter can be compactly formulated if one introduces Grassmann odd ghost variables  $b_\alpha$  with ghost number  $\text{gh}(b_\alpha) = -1$ . In terms

of generating functions  $\Psi(X, A|b)$  those constraints from (3.1) - (3.3) that do not involve  $N_i^j, N_i, N_X$  stay intact while the remaining ones take the form

$$(N_i^j + B_i^j)\Psi = 0 \quad i < j, \quad (N_i + B_i)\chi = s_i\Psi, \quad (N_X - B - w)\Psi = 0. \quad (3.10)$$

Here the following notation for ghost contributions have been introduced:

$$B_i^j = \delta_i^\alpha \delta_\beta^j b_\alpha \frac{\partial}{\partial b_\beta}, \quad B_\alpha = b_\alpha \frac{\partial}{\partial b_\alpha}, \quad B = \sum_{\alpha=1}^p B_\alpha. \quad (3.11)$$

It is easy to see that for zeroth ghost degree component  $\Psi^{(0)} = \phi(X, A)$  and for degree minus one component  $\Psi^{(-1)} = \chi(X, A|b) = \chi^\alpha(X, A)b_\alpha$  the constraints for fields and gauge parameter are reproduced. At the same time the gauge transformation takes the usual form

$$\delta\phi = \mathcal{Q}_p\chi, \quad \mathcal{Q}_p = \mathcal{S}_\alpha^\dagger \frac{\partial}{\partial b_\alpha}, \quad (3.12)$$

where  $\mathcal{Q}_p$  is a BRST operator,  $\text{gh}(\mathcal{Q}_p) = 1$ .

### 3.3 Interpretation of parameters

Our theory is determined by several parameters which are spins  $s_1 \geq s_2 \geq \dots \geq s_{n-1}$ , real parameter  $w$ , integer parameter  $p$  entering the formulation through the gauge equivalence (3.5) and constraints (3.6). In addition, for special values of  $w$  extra integer parameter  $t$ , the “depth” of gauge transformations also shows up through the constraint (3.8).

To see which representation we are dealing with let  $\Phi(X, A)$  represent an equivalence class of field configurations modulo the gauge equivalence generated by  $\mathcal{Q}_p$ , i.e.,  $\Phi \sim \Phi + \mathcal{Q}_p\chi$  with  $\chi = b_\alpha\chi^\alpha$ . We then explicitly evaluate the value of the quadratic Casimir operator

$$C_2 = -\frac{1}{2}J_{AB}J^{AB}, \quad J_{AB} = L_{AB} + M_{AB}, \quad (3.13)$$

where the orbital and the spin parts are given by

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}, \quad M_{AB} = \sum_{i=1}^{n-1} \left( A_{Ai} \frac{\partial}{\partial A_i^B} - A_{Bi} \frac{\partial}{\partial A_i^A} \right). \quad (3.14)$$

Taking into account constraints (3.1)-(3.3) a direct calculation yields

$$C_2\Phi = \left( w(w+d-1) + \sum_{l=1}^{n-1} s_l(s_l - 2l + d - 1) \right) \Phi - 2 \sum_{l=1}^{n-1} \mathcal{S}_l^\dagger \bar{\mathcal{S}}^l \Phi. \quad (3.15)$$

Let us analyze the last term in (3.15) in more detail. Summands  $\mathcal{S}_l^\dagger \bar{\mathcal{S}}^l$  with  $l > p$  vanish because of the constraints (3.6). The remaining ones are identically rewritten as

$$\sum_{\alpha=1}^p \mathcal{S}_\alpha^\dagger \bar{\mathcal{S}}^\alpha \Phi = \mathcal{Q}_p\chi, \quad \chi = b_\alpha \bar{\mathcal{S}}^\alpha \Phi. \quad (3.16)$$



It is easy to see that  $\chi$  satisfies all the necessary constraints provided  $\Phi$  does. Indeed, the only nontrivial point is to check that  $(N_i^j + B_i^j)\chi = 0$  but this follows from  $[N_i^j + B_i^j, b_\alpha \bar{S}^\alpha] = 0$ , which in turn is algebraically analogous to  $[N_i^j + B_i^j, S_\alpha^\dagger \frac{\partial}{\partial b_\alpha}] = 0$ . Note that for the above argument to work the entire set of oscillators  $A_i$  is split into two complementary parts: oscillators  $A_\alpha$ ,  $\alpha = 1, \dots, p$  involved in the gauge transformations  $\delta\Phi = S_\alpha^\dagger \chi^\alpha$  and oscillators  $A_{\hat{\alpha}}$ ,  $\hat{\alpha} = p+1, \dots, n-1$  entering tangent constraints  $\bar{S}^{\hat{\alpha}}\Phi = 0$ . In other words the respective Young tableau is cut into two complementary parts: the upper part subjects to the gauge equivalence and the lower part subjects to the tangent constraints.

It follows that the last term in (3.15) is pure gauge (cohomologically trivial) and does not contribute to the value of the Casimir operator understood as acting on equivalence classes of field configurations modulo gauge transformations. In a more refined language what we have just computed is the value of the second Casimir operator in the  $\mathcal{Q}_p$ -cohomology at zeroth ghost degree. This is a well-defined problem because  $C_2$  commutes with  $\mathcal{Q}_p$  as well as with all the constraints and hence acts in the cohomology.

All in all, one obtains

$$C_2\Phi = \left(w(d-1+w) + \sum_{l=1}^{n-1} s_l(s_l - 2l + d - 1)\right)\Phi, \quad (3.17)$$

so that the analysis in terms of gauge equivalence classes gives the same result as the gauge fixed analysis of [3, 7]. Again following [3, 7] we compare the obtained value with the known value of  $C_2$  in the representation with energy  $E_0$  and the same spin. This gives the following identification

$$E_0(E_0 - d + 1) = w(w + d - 1), \quad (3.18)$$

so that there are two possible energy values

$$E_0^1 = -w, \quad E_0^2 = w + d - 1. \quad (3.19)$$

Let us discuss two cases separately. If  $w$  is special, *i.e.*,  $w = s_p - p - t$  one gets  $E_0^1 = -(s_p - p - t)$  and  $E_0^2 = s_p - p - t + d - 1$ . According to Refs. [3, 49, 13] the correct value of the vacuum of mixed symmetry massless or partially massless fields is given by  $E_0^2$ .

If  $w$  is generic, the gauge symmetry can be shown purely algebraic so that there are no genuine gauge fields. After fixing this algebraic gauge symmetry one arrives at the formulation without gauge symmetry at all. The respective field  $\tilde{\Phi}(x, A)$  depending on intrinsic AdS coordinates  $x^m$ , where  $m = 0, \dots, d-1$ , satisfies the following equations of motion

$$\tilde{\square}\tilde{\Phi} = \mu^2\tilde{\Phi}, \quad \mu^2 = w(w + d - 1), \quad (3.20)$$

along with further differential and algebraic constraints originating from respectively  $S^i\Phi = 0$  and constraints (3.1). Here  $\tilde{\square}$  is an operator representing  $-\frac{1}{2}L_{AB}L^{AB}$  in terms of chosen

representatives of equivalence classes. Note that the explicit parameterization of  $\tilde{\Phi}$  and the explicit form of the  $\tilde{\square}$  and further constraints depend on the gauge choice and is discussed in more details in Section 6.

## 4 Generating BRST formulation

The formulation based on the ambient space is not manifestly local. Indeed, even if one explicitly solves the radial constraint and represents fields, constraints, and gauge transformations in terms of intrinsic coordinates on AdS space the gauge parameter is still subjected to the differential constraints (besides the purely algebraic ones). A possible way out is to use a BRST first-quantized technique and to impose the constraints involving  $X^A$ -derivatives through the BRST procedure by adding them to the “minimal” BRST operator  $\mathcal{Q}_p$  with their own ghost variables (see the discussion in Section 4.3). This extends the spectrum of fields and ensures that the gauge parameter is not subjected to differential constraints. Another, though equivalent, approach is to enlarge the space of fields in a more geometrical way by putting the ambient space to a fiber of a vector bundle over AdS space [28, 29, 34]. Here we follow the respective considerations in [24] and hence skip details.

### 4.1 Space of fields and BRST operator

A well-known and extremely useful way to describe AdS geometry<sup>6</sup> is to consider a trivial vector bundle  $V$  over  $d$ -dimensional AdS space with the fiber being the ambient space  $\mathbb{R}^{d-1,2}$  and the structure group  $O(d-1, 2)$ . Assume in addition that the bundle is equipped with the flat  $o(d-1, 2)$ -connection  $\omega_m^{AB}(x)$  and a fixed section  $V^A(x)$  satisfying  $\eta_{AB}V^AV^B = -1$ , where  $\eta_{AB}$  is the standard fiberwise pseudoeuclidean metric (2.1)<sup>7</sup>. If in addition, a local frame  $e_m^A(x) = \nabla_m V^A(x)$  has a maximal rank (*i.e.*,  $d$ ) at any point then  $\omega_m^{AB}(x)$ ,  $V^A(x)$  determine negative constant curvature geometry. Indeed,  $g_{kl} = \eta_{AB}e_k^Ae_l^B$  gives the AdS metric. Using a special local frame where  $V^A = (0, \dots, 0, 1)$  it is easy to observe that the flatness condition for  $\omega_m^{AB}$  reproduces the negative constant curvature condition for  $g_{kl}$ .

In addition we introduce space  $\mathcal{H}$  of polynomials in  $A_i^A$  and ghosts  $b_\alpha$  with coefficients in formal power series in variables  $Y^A$  and where the  $sp(2n)$  and  $o(d-1, 2)$  algebras are given in a twisted realization as explained in section 2.1.2. For the moment we do not take explicitly into account the fiber version of the constraints (3.1) - (3.3), and (3.8) because now they are purely algebraic and concentrate first on the relevant geometrical structures.

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<sup>6</sup>Analogous technique is easily extended to generic constant curvature and conformal spaces or, more generally, parabolic geometries. It is essentially a version of the well-known Cartan description.

<sup>7</sup>Section  $V^A$  plays the role of the compensator field ( see, *e.g.*, [50])

The vector bundle we are interested in is a vector bundle associated to  $V$  and with the fiber being  $\mathcal{H}$ . In particular, the flat connection  $\omega_m^{AB}$  determines a flat covariant derivative

$$\nabla = \theta^m \frac{\partial}{\partial x^m} + \frac{1}{2} \theta^m \omega_m^{AB} J_{AB}, \quad (4.1)$$

where  $J_{AB}$  are  $so(d-1, 2)$  generators (2.2) acting on  $\mathcal{H}$  in a twisted realization and we have assumed the local frame where  $V^A = \text{const.}$  Here and below we replace basis differentials  $dx^m$  on AdS space with the Grassmann odd ghost variables  $\theta^m$ ,  $m = 0, \dots, d-1$ ,  $\text{gh}(\theta^m) = 1$  because in the BRST formulation  $\nabla$  appears as a part of BRST operator.

The BRST extended space of states  $\mathcal{H}$  is given by differential forms of all ranks on AdS with values in the bundle. In plain terms they are  $\mathcal{H}$ -valued fields depending on  $x^m, \theta^m$ . Note that from the first-quantized point of view the space of field configurations is the BRST extended space of quantum states. The component fields entering  $\Psi = \Psi(x, \theta|Y, A, b)$  have the following structure

$$\Psi_{m_1 \dots m_r}{}^{A_1, \dots, A_l, \dots; \alpha_1 \dots \alpha_k}(x), \quad (4.2)$$

where  $A_l$  are  $o(d-1, 2)$  vector indices while  $\alpha_k$  and  $m_r$  are antisymmetric indices because the respective ghost variables  $b_\alpha$  and  $\theta^m$  are anticommuting.

On the space of states  $\mathcal{H}$  we define the following BRST operator

$$\hat{\Omega} = \nabla + Q_p. \quad (4.3)$$

Here  $\nabla$  is the covariant derivative (4.1) and  $Q_p$  is the algebraic operator given by

$$Q_p = S_\alpha^\dagger \frac{\partial}{\partial b_\alpha}, \quad (4.4)$$

where  $S_\alpha^\dagger$  are  $sp(2n)$  generators (2.9). Of course  $Q_p$  is precisely the fiber version of the ambient space BRST operators  $\mathcal{Q}_p$  from (3.12).

Because of the ghost degree prescription  $\text{gh}(\theta^m) = -\text{gh}(b_\alpha) = 1$  BRST operator  $\hat{\Omega}$  has a standard ghost degree  $\text{gh}(\hat{\Omega}) = 1$ . Moreover, it follows from  $\nabla^2 = 0$ ,  $Q_p^2 = 0$  and  $o(d-1, 2) - sp(2n)$ -bimodule structure according to which  $J_{AB}$  commutes with all the  $sp(2n)$  generators that  $\hat{\Omega}$  is nilpotent so that it can be consistently interpreted as a BRST operator.

## 4.2 Fiber constraints and equations of motion

Before discussing equations of motion and gauge symmetries we need to impose the fiber version of the constraints introduced in the ambient space description of Section 3. More precisely, off-shell constraints

$$T^{ij}\Psi = 0, \quad (N_i^j + B_i^j)\Psi = 0 \quad i < j, \quad (N_i + B_i)\Psi = s_i\Psi, \quad (4.5)$$

stay the same while those involving  $A_0^A$  (i.e., (3.3) and (3.2), (3.6)) take the form

$$\square_Y \Psi = 0, \quad S^i \Psi = 0, \quad (4.6)$$

and

$$h \Psi = 0, \quad h = N_Y - B - w, \quad (4.7)$$

$$\bar{S}^{\hat{\alpha}} \Psi = 0, \quad \hat{\alpha} = p + 1, \dots, n - 1. \quad (4.8)$$

Here we recall that  $N_Y = (Y^A + V^A) \frac{\partial}{\partial Y^A}$  and  $\bar{S}^i = (Y^A + V^A) \frac{\partial}{\partial A_i^A}$  to stress the difference with the ambient space realization. For special values  $w = s_p - p - t$  one in addition imposes the fiber version of (3.8):

$$(\bar{S}^p)^t \Psi = 0. \quad (4.9)$$

Note that the trace constraints in (4.5) and (4.6) can be collectively written as  $T^{IJ} \Psi = 0$ .

That operator  $Q_p$  acts in the subspace singled out by the above constraints follows from the constraint algebra which is identical to the one of the ambient space description. Covariant derivative  $\nabla$  commutes with all the constraints because of  $o(d - 1, 2) - sp(2n)$  bimodule structure.

According to the general prescription the physical fields <sup>8</sup> are identified as elements  $\Psi^{(0)}$  at ghost number 0 and gauge parameters as elements  $\Psi^{(-1)}$  at ghost number  $-1$  (see, e.g., [28, 31]). Their component form read off from (4.2) is given by  $k - l = 0$  and  $k - l = -1$ , respectively. The equations of motion and the gauge transformations read as

$$\hat{\Omega} \Psi^{(0)} = 0, \quad \delta \Psi^{(0)} = \hat{\Omega} \Psi^{(-1)}. \quad (4.10)$$

The component form of these equations was given in [24]. Reducibility gauge parameters are described by ghost-number  $-n$  elements and the respective transformations read as  $\delta \Psi^{(-n)} = \hat{\Omega} \Psi^{(-n-1)}$ . Elements of positive ghost degree correspond to the equations of motion and their (higher) reducibility relations.

Let us comment on the use of the BRST approach in the present context. Usually the BRST operator is assumed to be hermitian with respect to the inner product in the representations space. In this case the equations of motion of the associated free field theory can be derived from a local action of the form  $\langle \Psi^{(0)}, \Omega \Psi^{(0)} \rangle$ . Throughout this paper we do not require existence of an inner product and the hermiticity of the BRST operator. From the field theory point of view this corresponds to working at the level of equations of motion and their gauge symmetries. This approach was described [28] to which we refer for further details.

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<sup>8</sup>Sometimes the term “physical” is used to denote a minimal covariant field content usually obtained by eliminating auxiliary fields and Stueckelberg variables.

### 4.3 Parent formulation

Although the constructed formulation is very compact it is important to stress that the representation space is highly constrained. A description where (almost) all the constraints are implemented through the BRST procedure so that the space of fields is (nearly) unconstrained can be useful. Formulations of this type are known as parent ones and can be used to generate other formulations through the elimination of generalized auxiliary fields.

Here we briefly discuss a version of the parent formulation generalizing the one from [24] and where all the constraints involving derivatives with respect to  $Y^A$  are implemented through a BRST procedure. Namely, parent BRST operator reads as

$$\Omega^{\text{parent}} = \nabla + \bar{\Omega}, \quad \bar{\Omega} = Q_p + \text{“more”} = S_\alpha^\dagger \frac{\partial}{\partial b_\alpha} + c_0 \square + c_i S^i - \delta_\alpha^i c_i \frac{\partial}{\partial b_\alpha} \frac{\partial}{\partial c_0}, \quad (4.11)$$

where new Grassmann odd ghost variables  $c_0$  and  $c_i$  have been introduced. Note that the remaining constraints (4.5) and (4.7)-(4.9) or, more precisely, their  $\bar{\Omega}$ -invariant extensions are still imposed directly.

Using this form of the theory one can easily obtain the proper ambient space BRST description where in contrast to the formulation of Section 3 gauge parameters are not subjected to differential constraints. Indeed, following [24] one shows that the parent theory is equivalent to the ambient space BRST formulation determined by  $\bar{\Omega}$  where all the constraints are taken in the ambient space realization of 2.1.1 instead of the twisted one from section 2.1.2. The same applies to off-shell constraints (4.7)-(4.9).<sup>9</sup>

Implementing the remaining constraints through the BRST operator can be easily performed in the particular case of unitary massless fields, *i.e.*, where  $s_1 = \dots = s_p$ . In this case  $\mathcal{N}_\alpha^\beta \Psi = 0$  for all  $\alpha, \beta$  so that  $\bar{S}^p \Psi = 0$  imply  $\bar{S}^\alpha \Psi = 0$ . Consider the following BRST operator

$$\bar{\Omega}^{\text{tot}} = Q_p + c_{IJ} T^{IJ} + \nu_i \bar{S}^i + \mu h + \gamma_j^i N_i^j + \text{ghost terms}, \quad (4.12)$$

where we have introduced ghost variables  $c_{IJ}$ ,  $\nu_i$ , and  $\gamma_j^i$   $i < j$  associated to constraints  $T^{IJ}$ ,  $\bar{S}^i$  and  $N_i^j$   $i < j$ . It turns out that the parent theory based on  $\bar{\Omega}^{\text{tot}}$  is equivalent to the starting point formulation based on  $Q_p$  and constraints (4.5)-(4.9). The proof is given in Appendix A.

A few comments are in order. Note that  $\bar{\Omega}^{\text{tot}}$ -invariant extension of the remaining constraints  $(\mathcal{N}_i - s_i)\Psi = 0$  are imposed directly in the representation space. We do not add these constraints to the BRST operator with their own ghosts because this in general leads to extra cohomology classes. However, imposing them directly is not a real problem

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<sup>9</sup>It is worth mentioning that the structure of the resulting BRST operator which is just  $\bar{\Omega}$  (4.11) realized differently is very similar to the BRST operator used to describe bosonic strings and HS fields on the Minkowski space (see, *e.g.*, [23, 28, 31]). We expect this formulation to be useful in constructing the respective Lagrangian description and analyzing the spectrum of strings on AdS.

because the entire representation space decomposes into the direct sum of eigenspaces associated to different values of  $s_i$  and the BRST operator preserves the decomposition. This makes the space subjected to BRST invariant extensions of  $(\mathcal{N}_i - s_i)\Psi = 0$  almost as convenient as a totally unconstrained space.

As far as the general case is concerned the above arguments are not immediately applicable. This means that using the appropriate generalization of (4.12) can, in principle, bring extra fields and hence spoil the equivalence. Extending (4.12) beyond the unitary case remains an open problem.

## 5 $Q_p$ -cohomology analysis

For a system whose BRST operator has the structure  $\Omega = \nabla + Q$  with  $Q$  algebraic an important information is encoded in the  $Q$ -cohomology. In the case at hand the relevant cohomology is the  $Q_p$ -cohomology in the fiber, *i.e.*, the subspace of  $\mathcal{H}_{\text{on-shell}} \subset \mathcal{H}$  determined by constraints (4.5)-(4.9). The  $Q_p$ -cohomology is graded by ghost number (note that ghosts  $\theta^m$  are not the fiber variables and hence do not contribute to ghost degree in  $\mathcal{H}$ ).

The  $Q_p$ -cohomology can be given various interpretations. First of all, eliminating all the generalized auxiliary fields associated to elements that are not in the cohomology one reduces the system to the form where fields take values in  $Q_p$ -cohomology only. Such a formulation, known as unfolded formulation,<sup>10</sup> is in some sense minimal among the formulations where the space-time derivatives enter only through the de Rham differential. Elements of  $Q_p$ -cohomology at ghost degree  $-k$  give rise to physical fields which are  $k$ -forms, gauge parameters which are  $k - 1$ -forms, etc. In particular, in the context of unfolded approach  $Q_p$ -cohomology at vanishing ghost degree is known as Weyl module<sup>11</sup> while those at negative degree as a gauge module. Their associated fields are 0 and  $k$ -forms and can be related to respectively linearized curvatures and gauge fields.

The analogue of  $Q_p$ -cohomology can be identified for a general gauge theory as well. Starting from a Batalin–Vilkovisky formulation of a given gauge theory in jet space terms (see, *e.g.*, [52] and references therein) and restricting to the stationary surface (by eliminating contractible pairs for Koszul–Tate part  $\delta$  of the BRST differential) one ends up with the formulation based on gauge part  $\gamma$  of the BRST differential. Finally, eliminating all the contractible variables for  $\gamma$  one reduces the system to the form where the reduced  $\gamma$  is at least quadratic. The remaining variables are the generalized tensor fields and connections of [53], where, in particular, the reduced  $\gamma$  has been explicitly computed for various gauge models including Yang–Mills theory and Einstein gravity. It turns out that in the

<sup>10</sup>Note that the unfolded approach [51] was originally developed from a different perspective.

<sup>11</sup>Here, the term “module” refers to a space-time symmetry algebra that is  $o(d - 1, 2)$  in the present case.  $Q_p$ -cohomology is an  $o(d - 1, 2)$ -module because  $Q_p$  commutes with  $o(d - 1, 2)$  algebra or, more generally, with a space-time symmetry algebra in question.

case of linear theory these variables can be identified with the  $Q_p$ -cohomology. For instance, generalized tensor fields are associated to basis elements of the Weyl module while generalized connections to those of the gauge module. In the case of linear theories this relationship between the unfolded and the BRST approaches was established in [28, 29]. The case of general gauge theories was considered in [54, 55] to which we refer for the exhaustive discussion.

In this section we explicitly compute  $Q_p$ -cohomology in the subspace  $\mathcal{H}_{\text{on-shell}} \subset \mathcal{H}$  determined by (4.5) - (4.9). Namely, we show that

$$H^k(Q_p, \mathcal{H}_{\text{on-shell}}) = \begin{cases} \text{Weyl module} , & k = 0 , \\ 0 , & k \neq 0, -p , \\ \text{Gauge module} , & k = -p . \end{cases} \quad (5.1)$$

where  $H^k(\cdot)$  denotes cohomology at ghost degree  $k$ . The cohomology for intermediate ghost numbers is empty while for ghost number  $k = 0$  it is non-vanishing and can be identified with infinite-dimensional Weyl module. Cohomology at ghost degree  $-p$  is nonvanishing for special values (3.7) of parameter  $w$  only. In this case it describes the finite-dimensional gauge module introduced within the unfolded formulation [9, 10, 56, 12, 13, 24, 47].

The unfolded equations are determined by the BRST operator induced by  $\widehat{\Omega}$  from (4.3) in the cohomology of its second term  $Q_p$ . This can be computed using the standard homological technique as explained in [28, 29, 31]. We do not discuss the unfolded form of the equations of motion and gauge symmetries in further details and refer instead to [12, 13, 57].

The rest of this Section is devoted to the analysis of  $Q_p$ -cohomology in the case where parameter  $w$  take special values  $w = s_p - p - t$  with  $t = 1, 2, \dots, t_{\text{max}}$ . The case of generic  $w$  corresponds to massive field and is analyzed in Section 6.

## 5.1 Minimal ghost degree cohomology: gauge module

The coboundary condition is trivial at minimal ghost degree and therefore representatives of  $Q_p$ -cohomology at ghost degree  $-p$  are defined by the following constraints

$$\begin{aligned} S_\alpha^\dagger \Psi = 0 , \quad (\bar{S}^p)^t \Psi = 0 , \quad \bar{S}^{\widehat{\alpha}} \Psi = 0 , \quad (N_Y + t - s_p) \Psi = 0 , \\ \alpha = 1, \dots, p , \quad \widehat{\alpha} = p + 1, \dots, n - 1 , \end{aligned} \quad (5.2)$$

along with constraints (4.5). It is useful to describe solutions to (5.2) using the parameterization in terms of  $Y'^A = Y^A + V^A$ . This change of variables is legitimate because the first condition in (5.2) implies that for a homogeneous component the degree in  $Y^A$  cannot exceed that in  $A_i^A$  and hence  $\Psi$  is a finite order polynomial in  $Y^A$ .

Taking into account the total ghost degree a physical field associated to  $Q_p$ -cohomology at ghost degree  $-p$  is a differential  $p$ -form. This is because there should be exactly  $p$  variables  $\theta^m$  to gain a zeroth total ghost degree. This  $p$ -form on AdS space takes values in finite-dimensional irreducible  $o(d-1, 2)$ -module described by the Young diagram with the following lengths of rows

$$s_1 - 1 \geq \dots \geq s_{p-1} - 1 \geq s_p - 1 \geq s_p - t \geq s_{p+1} \geq \dots \geq s_{n-1} . \quad (5.3)$$

Note the row of length  $s_p - 1$  in the middle of a diagram with a subsequent row of a length  $s_p - t$ . According to [13] such fields describe spin  $(s_1, \dots, s_{n-1})$  partially massless AdS fields with the gauge symmetry associated to  $p$ -th row and having depth  $t$ .

## 5.2 Vanishing of intermediate ghost number cohomology

The proof that  $Q_p$ -cohomology vanishes at intermediate ghost numbers is based on the following observation: a representative of  $Q_p$  cohomology class of intermediate ghost degree can always be assumed polynomial. This can be shown by using cohomological arguments starting from the parent BRST operator (see Section 4.3) implementing all the constraints  $S_\alpha^\dagger, N_i^j, h, \bar{S}^{\hat{\alpha}}, (\bar{S}^p)^t$  with their own ghosts so that only the weight conditions are imposed directly. This reformulates the cohomological problem in the nearly unconstrained space. Note that one needs to keep in mind that in general this can bring extra cohomology classes. However, this does not affect the argument.

Using a suitable degree one then reduces the problem to the cohomology of the term implementing the constraints  $N_\alpha^\beta, S_\alpha^\dagger$  associated to the upper-half of the respective Young tableaux. The cohomology of the respective terms are known [31] and relevant representatives can be chosen polynomial. One then shows that completion of such elements to representatives of the total BRST operator can be also taken polynomial so that representatives can indeed be assumed polynomial. Another way to see that representatives can be assumed polynomial is to perform a direct analysis of the respective cocycle condition using the algebraic technique developed in [24].

In the space of polynomials it is then legitimate to use a new variable  $Y'^A = Y^A + V^A$  and hence to reformulate the problem as that of standard finite-dimensional  $sl(n)$ -modules. In this way one finds that for a polynomial element of ghost degree  $-k$  constraints (4.7) and (4.9) are in general inconsistent. Indeed, we obtain weight conditions

$$N_Y \Psi_k = (s_p + k - t - p) \Psi_k , \quad (N_p + B_p) \Psi_k = s_p \Psi_k , \quad (5.4)$$

along with

$$(\bar{S}^p)^t \Psi_k = 0 . \quad (5.5)$$

The last condition tells us that  $\#Y'^A \geq \#A_p^A - t + 1$  and this contradicts (5.4) except for  $k = p$  because  $\#A_p^A$  is either  $s_p$  or  $s_p - 1$  depending on whether ghost  $b_p$  is present or not. In this way we arrive at



**Proposition 5.1.** *The  $Q_p$ -cohomology evaluated in the subspace singled out by constraints (4.7) and (4.9) is empty in the ghost numbers  $0 < -k < p$ .*

In the space of formal power series there is also nontrivial cohomology at ghost degree 0. This is the Weyl module which we describe in the next Section.

### 5.3 Cohomology at vanishing ghost degree: Weyl module

The structure of the Weyl module for unitary massless AdS gauge fields was described in [12, 24] (see also [10] for early analysis and [8, 9] for the case of totally symmetric fields) and then for the general case involving partially massless and non-unitary massless fields in [47]. Just like in the case of unitary fields the generating BRST formulation gives an independent definition of the Weyl module as  $Q_p$ -cohomology at zeroth ghost degree. In this way the module structure is implemented in the construction from the very beginning because  $Q_p$  is  $o(d-1, 2)$ -invariant. Moreover, because the cocycle condition is trivial in this case the Weyl module is just a quotient of the  $o(d-1, 2)$ -module  $\mathcal{H}_{\text{on-shell}}$  modulo the  $o(d-1, 2)$ -invariant subspace. Recall that  $o(d-1, 2)$  algebra is realized in the twisted form, see section 2.1.2.

#### 5.3.1 Lorentz covariant basis

We choose a local frame where  $V^A = \delta_d^A$ . Set  $Y^a = y^a$  and  $Y^d = z$ . Analogously,  $A_i^a = a_i^a$  and  $A_i^d = u_i$ . In what follows, we always assume that all elements  $\Psi = \Psi(Y, A)$  are totally traceless,  $T^{IJ}\Psi = 0$ . Lemma B.1 formulated in Appendix B shows how constraints  $\bar{S}^i$  and  $h$  fix one or another type of dependence on  $(d+1)$ -th variables  $z$  and  $u_i$ . In particular, using lemma B.1 one can represent elements satisfying (4.7)-(4.9) as series in  $u_\alpha$  variables

$$\psi = \sum_{k \geq 0} u_{\alpha_1} \dots u_{\alpha_k} \psi_k^{\alpha_1 \dots \alpha_k}(a, y|b) . \quad (5.6)$$

The above series terminates at some finite order defined by spins  $s_\alpha$  and depth  $t$ . It follows that elements  $\psi$  do not depend on  $u_{\hat{\alpha}}$  and  $(u_p)^{t+m}$  for  $m \geq 0$ . In addition, homogeneity in  $u_\alpha$  gives a useful degree called level.

Both BRST operator  $Q_p$  and the constraints (4.5), (4.6) can be rewritten in terms of parameterization (5.6). In so doing the trace constraints remain unchanged while the weight and Young symmetry conditions take the form

$$(n_\alpha + u_\alpha \frac{\partial}{\partial u_\alpha} + B_\alpha - s_\alpha)\psi = 0 , \quad (n_{\hat{\alpha}} - s_{\hat{\alpha}})\psi = 0 , \quad (5.7)$$

$$(n_\alpha^\beta + u_\alpha \frac{\partial}{\partial u_\beta} + B_\alpha^\beta)\psi = 0 , \quad n_{\hat{\alpha}}^{\hat{\beta}}\psi = 0 , \quad (n_\alpha^{\hat{\beta}} - u_\alpha \bar{s}^{\hat{\beta}})\psi = 0 , \quad (5.8)$$

where  $\widehat{s}^\beta = y^a \frac{\partial}{\partial a_\beta^a}$ , and  $\alpha < \beta$  and  $\widehat{\alpha} < \widehat{\beta}$ . Using then constraint (4.7) rewritten in Lorentz terms as  $((z+1)\frac{\partial}{\partial z} + y^a \frac{\partial}{\partial y^a} - B + p + t - s_p)\phi = 0$  allows one to cast BRST operator into the following form

$$\widetilde{Q}_p = q_p - \widehat{h} u_\alpha \frac{\partial}{\partial b_\alpha}, \quad (5.9)$$

where  $\widetilde{Q}_p$  is operator (4.4) rewritten in terms of parameterization (5.6) and

$$q_p = s_\alpha^\dagger \frac{\partial}{\partial b_\alpha} \equiv a_\alpha^a \frac{\partial}{\partial y^a} \frac{\partial}{\partial b_\alpha}, \quad \widehat{h} = n_y - B + p + t - s_p. \quad (5.10)$$

Recall that  $(u_p)^t$  is zero in our subspace and therefore the respective contribution in (5.9) can also be vanishing. In particular, for  $t = 1$  the term in  $Q_p$  proportional to  $u_p$  vanishes. Note also that for unitary massless fields all  $u_\alpha = 0$  as a consequence of  $s_1 = \dots = s_p$  so that the reduced operator is simply  $q_p$  [24].

### 5.3.2 Weyl module

First of all we recall that a Poincaré Weyl (PW) module of spin  $l_1 \geq l_2 \dots \geq l_{n-1}$  [8, 58] can be defined [31] as a subspace of  $sl(n)$  HW vectors in the space of polynomials in  $y^a$  and  $a_i^a$  variables satisfying the respective weight conditions. One can view a PW module of spin  $l_1 \geq l_2 \dots \geq l_{n-1}$  as a subspace singled out by  $n_i^j \widetilde{\psi} = 0$ , remaining HW conditions  $\widetilde{s}_i \widetilde{\psi} = 0$ , weight conditions  $(n_i - l_i) \widetilde{\psi} = 0$ , and vanishing ghost degree condition  $\text{gh}(\widetilde{\psi}) = 0$ . Given AdS spin  $s_1 \geq \dots \geq s_{n-1}$  a PW module is called admissible associated if its weights  $l_i$  satisfy  $l_i = s_i - \nu_i$  where  $\nu_i = 0$ ,  $i \leq p$  and  $\nu_i \geq 0$ ,  $i > p$  and  $\nu_{p+1} + \dots + \nu_{n-1} \leq s_{p+1}$ .

For unitary fields the AdS Weyl module is isomorphic to a direct sum of admissible associated PW modules. The following Proposition is a slight generalization of this result. It turns out that  $H^0(q_p)$  calculated for spin weights  $(m_1, \dots, m_{n-1})$  and denoted by  $\mathcal{M}_{0,p,m}$  can be decomposed into a direct sum of some PW modules.

**Proposition 5.2.** *The zero-ghost-number cohomology  $\mathcal{M}_{0,p,m}$  of BRST operator  $q_p$  evaluated in the subspace (5.7), (5.8) is isomorphic to a direct sum of admissible associated PW modules.*

More detailed discussion of the above proposition is relegated to Appendix B. Spin weights  $\{m\}$  of admissible PW modules are defined by original spins and parameters  $p$  and  $t$  through weight constraints (5.7). Denoting  $H^0(q_p)$  on the  $k$ -th level (see (5.6)) as  $\mathcal{M}_{0,m,p}^{(k)}$  and its spin weights as  $\{m\}_k$  we find that spins are given by  $m_{\widehat{\alpha}} = s_{\widehat{\alpha}}$  for  $\widehat{\alpha} = p+1, \dots, n-1$ , and  $m_\alpha = s_\alpha - k_\alpha$  for  $\alpha = 1, \dots, p$  such that  $k_1 + \dots + k_p = k$ .

Computation of  $Q_p$ -cohomology reduces then to inspecting how the second term in (5.9) acts in  $H^0(q_p)$ . One can show that using this term any level- $k$  element  $\psi_k \in \mathcal{M}_{0,l,p}$  whose degree in  $y^a$  is smaller than  $s_1$  can be set to zero. Denoting the subspace of all

such elements from  $\mathcal{M}_{0,m,p}^{(k)}$  as  $\mathcal{Z}_{0,m,p}^{(k)}$  we arrive at the component description of AdS Weyl cohomology (see Appendix **B** for more details).

**Proposition 5.3.** *AdS Weyl module  $\mathcal{M}_0$  of a given spin is isomorphic to a direct sum of quotient spaces*

$$\mathcal{M}_0 = \bigoplus_{k \geq 0} \bigoplus_{\{m\}_k} \mathcal{M}_{0,m,p}^{(k)} / \mathcal{Z}_{0,m,p}^{(k)} , \quad (5.11)$$

where  $\{m\}_k$  denotes a set of admissible spin weights on the  $k$ -th level.

For unitary fields operator  $\hat{h}$  does not contribute and  $Q_p = q_p$ . As a result  $\mathcal{Z}_{0,m,p}^{(k)} = 0$  and AdS Weyl cohomology is a direct sum of admissible PW modules. In other words, we reproduce here the Brink-Metsaev-Vasiliev conjecture put forward in [40] and proved in [12, 24, 47]. For non-unitary fields AdS Weyl module is not a direct sum of admissible PW modules. As an illustration in Appendix **B** we perform the analysis in the particular case of partially massless totally symmetric fields, *i.e.* when  $n = 2$ ,  $p = 1$ ,  $t \geq 1$ .

## 6 Massive fields

We now consider the case of generic values of  $w$ . A crucial observation is that in this case the gauge symmetry determined by  $Q_p$  is purely algebraic. It implies that the theory is equivalent to the one without gauge freedom through the elimination of generalized auxiliary fields. The approach taken in this Section is an extension of that from [32] to the case of mixed symmetry fields. Mention the related considerations in [12], where the algebraic nature of the gauge invariance in the massive case was observed within the unfolded framework.

The essential step is the following

**Proposition 6.1.** *For  $w$  generic the  $Q_p$ -cohomology in the space of elements  $\Psi(Y, A, b_\alpha)$  satisfying (4.5)-(4.8) can be identified with  $b_\alpha$ -independent elements satisfying in addition  $\bar{S}^\alpha \Psi = 0$  so that the entire set of constraints reads as*

$$T^{IJ} \Psi = 0, \quad \bar{S}^i \Psi = 0, \quad N_i^j \Psi = 0 \quad i < j, \quad h \Psi = 0, \quad (N_i - s_i) \Psi = 0. \quad (6.1)$$

It follows from the Proposition that after reducing to  $Q_p$ -cohomology there are no elements of negative ghost degree left and hence no gauge fields.

*Proof.* Let us consider first  $Q_p$ -cohomology in the space of elements  $\Psi(Y, A, b_\alpha)$  satisfying  $h \Psi = 0$  only. Using Lemma **B.1** in the sector of  $z$  variables only one finds that this space is isomorphic to the subspace of  $z$ -independent elements and the isomorphism map amounts to simply putting  $z$  to zero. Its inverse is constructed recursively order by order in  $z$  (see [29, 24] for more details). In terms of  $z$ -independent subspace  $Q_p$  is represented by  $\tilde{Q}_p$  given by

$$\tilde{Q}_p = q_p + \hat{h} u_\alpha \frac{\partial}{\partial b_\alpha}, \quad (6.2)$$

where as before  $q_p = a_\alpha^a \frac{\partial}{\partial y^a} \frac{\partial}{\partial b_\alpha}$  and  $\widehat{h} = n_y - B - w$ . In contrast to the case of special  $w$  the coefficient in front of the second term never vanishes if  $w$  is generic. Using a suitable degree one reduces the problem to the cohomology of the second term which, in turn, is isomorphic to  $u_\alpha, b_\alpha$ -independent elements. In fact the reduced BRST operator vanishes in this case because there are no more ghost variables left so that  $\widetilde{Q}_p$  cohomology can be identified with  $u_\alpha, b_\alpha$ -independent elements. Restoring  $z$ -dependence the  $Q_p$ -cohomology can be identified as the subspace  $\frac{\partial}{\partial u_\alpha} \Psi = 0$ ,  $h\Psi = 0$ , and  $\frac{\partial}{\partial b_\alpha} \Psi = 0$ . Note that any  $Q_p$ -cocycle vanishing at  $u_\alpha = b_\alpha = 0$  is trivial.

The following identification of the above cohomology as a subspace is more useful

$$\bar{S}^\alpha \Psi = 0, \quad h\Psi = 0, \quad \frac{\partial}{\partial b_\alpha} \Psi = 0. \quad (6.3)$$

The two spaces are clearly isomorphic as can be seen by using a version of Lemma **B.1** in the sector of  $u_\alpha$  variables. Moreover, if  $\psi$  and  $\psi'$  satisfy respectively  $\frac{\partial}{\partial u_\alpha} \psi = h\psi = 0$  and  $\bar{S}^\alpha \psi = h\psi = 0$  along with  $\psi_{u_\alpha=0} = \psi'_{u_\alpha=0}$  then  $\psi - \psi'$  is a coboundary. Indeed, the difference  $(\psi - \psi')$  vanishes at  $u_\alpha = 0$  and hence is trivial in  $Q_p$ -cohomology.

In order to take other constraints into account one starts with the parent BRST operator implementing all the constraints

$$S_\alpha^\dagger, \quad \bar{S}^\alpha, \quad N_i^j \quad i < j, \quad (6.4)$$

with their own ghost variables and acting in the subspace of elements satisfying  $h'\psi = (N'_i + B_i - s_i)\psi = 0$ , where  $h', N'_i$  are operators  $h, N_i$  modified by necessary contributions from the extra ghost variables. Note that in the representation space the ghost variables associated to extra constraints are represented by coordinate ghosts carrying positive ghost degree in contrast to momenta ghosts  $b_\alpha$  representing ghosts associated to gauge generators. Although all representations of fermionic ghosts are equivalent this is a standard choice if the ghosts degree in the representation space is normalized such that  $\text{gh}(1) = 0$  and physical fields appear at zeroth ghost degree. As we have already seen the parent reformulation can in general bring extra cohomology classes but they do not affect the argument.

Using a suitable degree such that  $Q_p$  is the lowest degree term of the total BRST operator one reduces the cohomology problem to the  $Q_p$ -cohomology identified as a subspace (6.3) and then considers the reduced cocycle condition for an element  $\Psi$  of vanishing ghost degree. Taking into account that  $\Psi$  is necessarily ghost-independent (because  $b_\alpha$  are eliminated there are no variables of negative ghost degree left) one indeed finds all the constraints (6.1) besides the trace constraints  $T^{IJ}\Psi = 0$ . That the same analysis remains true in the totally traceless subspace can be seen using the cohomological arguments from [29, 24].  $\square$

The statement can be rephrased by saying that for generic values of  $w$  the gauge invariance determined by  $Q_p$  is purely algebraic and that  $\bar{S}^\alpha \Psi = 0$  is a proper gauge condition

completely removing the gauge freedom. Indeed, if  $\chi^\alpha$  is a gauge parameter satisfying (4.5)-(4.8) then  $\bar{S}^\beta S^\dagger_\alpha \chi^\alpha = 0$  implies  $\chi^\alpha = 0$ . In this gauge the equations of motion are simply constraints (6.1) along with the  $\nabla\Psi = 0$  where now  $\Psi$  is a zero form on AdS space. In particular, constraints (6.1) give an  $o(d-1, 2)$  covariant definition of Weyl module in the massive case.<sup>12</sup> Let us note that the above statement do not directly apply if  $w$  is integer but not special (more precisely, such that  $\hat{h}$  is not invertible). In this case there are still no genuine gauge fields but the structure of the Weyl module can be different (see the respective discussion in [32]).

We now turn back to the formulation in terms of the ambient space. Consider the system determined by (6.1) where the constraints are realized on ambient space functions  $\Psi(X, A)$  instead of the fiber ones  $\Psi(Y, A)$ . Using the arguments from [24] one can show that the system is equivalent<sup>13</sup> to the above system on the AdS space. In particular, this shows that  $\bar{S}^\alpha \Psi = 0$  are again proper gauge conditions.

To obtain the explicit form of equations of motion in terms of tensor fields on the hyperboloid one employs the standard isomorphism between ambient space tensor fields satisfying  $(N_X - w)\Psi = 0$  along with  $\bar{S}^i \Psi = 0$  and respective tensor fields on the hyperboloid (see, *e.g.*, [7]). More explicitly such an ambient space field  $\Psi = \Psi(X, A)$  gives rise to AdS tensor field  $\psi(x, a_i)$  according to  $\psi(x, a_i) = \Psi(X^A(x), \frac{\partial X^A}{\partial x^m} a_i^m)$ , where  $X^A(x)$  describe the embedding in terms of local coordinates on the hyperboloid.

Under the isomorphism the ambient space operator  $\frac{\partial}{\partial X^A} - \frac{X_A X^B}{X^2} \frac{\partial}{\partial X^B}$  is mapped to the Levi-Civita covariant derivatives on AdS tensor fields (see, *e.g.*, [7, 48]). Using the isomorphism the ambient space constraints  $\bar{S}^i \Psi = 0$ ,  $\square_X \Psi = 0$ ,  $(N_X - w)\Psi = 0$  indeed give rise to usual massive equations of motion<sup>14</sup>

$$(g^{kl} \nabla_k \nabla_l + \sum_{i=1}^{n-1} s_i) \psi = w(w + d - 1) \psi, \quad \nabla^m \frac{\partial}{\partial a_i^m} \psi = 0, \quad (6.5)$$

where  $g^{kl}$  is the inverse to the AdS metric  $g_{kl} = \eta_{AB} \frac{\partial X^A}{\partial x^k} \frac{\partial X^B}{\partial x^l}$ . At the same time the algebraic constraints  $T^{ij} \Psi = 0$ ,  $(N_i - s_i) \Psi = 0$ ,  $N_i{}^j \Psi = 0$   $i < j$  remain unchanged except one needs to rewrite them in terms of  $\psi(a_i)$  and the AdS metric. Note that constraints  $h\Psi = \bar{S}^i \Psi = 0$  are needed for the isomorphism and do not produce any conditions on AdS tensor fields.

<sup>12</sup>In the case of totally symmetric fields the respective Weyl module and the unfolded formulation were originally studied in [57] within a different framework.

<sup>13</sup>More precisely, one needs the arguments given in Section 3.3 of [24] restricted to the case where no gauge freedom is present. The idea is to reformulate the ambient space theory in the ambient parent form and then pull-back the covariant derivative to the hyperboloid.

<sup>14</sup>Here we make use of formulas from [7] relating the ambient and the intrinsic Laplacians.

## 7 Conclusions

In this paper we have constructed the unified BRST formulation for arbitrary bosonic fields in  $AdS_d$  spacetime. The space of field configurations is identified as a subspace in the ambient configuration space which is naturally an  $o(d-1, 2) - sp(2n)$  bimodule. A set of particular constraints needed to describe a given AdS field depends on its spin weights, the vacuum energy, and the depth of its (partially massless) gauge invariance. The ambient space formulation for massless fields successfully reproduces the results of Metsaev [3], while for partially massless and massive fields the proposed set of fields, their gauge symmetries, and equations of motion are new.

In addition to ambient space description an explicitly local generating BRST formulation is constructed by, roughly speaking, treating the ambient space as a fiber of a vector bundle over the AdS space-time. In this case the  $o(d-1, 2) - sp(2n)$  bimodule structure is realized on the fiber in a twisted way which is in contrast to the standard realization employed in the ambient space description. The twisted realization is essential for the entire construction and can be regarded as a twisted version of Howe duality.

It is important to stress the role played by BRST operator  $Q_p$  associated to the particular  $sp(2n)$  basis elements. It encodes a gauge symmetry of the theory, both differential and algebraic in the ambient space formulation, and pure algebraic in the generating BRST formulation. In the later case we show that non-empty  $Q_p$ -cohomology is identified with the generalized curvatures (Weyl module) and the generalized connections (gauge module) of the unfolded formulation [9, 10, 11, 12, 13] reproducing the set of unfolded fields. The full system of unfolded equations can be explicitly determined by the BRST differential reduced to  $Q_p$ -cohomology.

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## A BRST Cohomology associated with $sl(n)$ -modules and the equivalence proof.

We are interested in the cohomology of the BRST operator of the upper-triangular subalgebra of  $sl(n-1)$  formed by  $N_i^j, i < j$  (2.4) with coefficients in a given finite-dimensional

representation. Introducing ghost variables  $\gamma_j^i$ ,  $i < j$  the BRST operator reads as

$$\Omega = \gamma_j^i N_i^j - \gamma_l^i \gamma_j^l \frac{\partial}{\partial \gamma_j^i}. \quad (\text{A.1})$$

It is defined on the tensor product of an  $sl(n-1)$ -module with a Grassmann algebra generated by ghosts  $\gamma_j^i$ . Restricting operator (A.1) to a subspace of elements with definite  $sl(n-1)$  weights we introduce the following BRST extension of elements (2.4):

$$\hat{N}_i = N_i - \gamma_k^i \frac{\partial}{\partial \gamma_k^i} + \gamma_i^k \frac{\partial}{\partial \gamma_i^k}. \quad (\text{A.2})$$

It is then easy to check that the following subspace

$$(\hat{N}_i - s_i)\phi = 0 \quad (\text{A.3})$$

is  $\Omega$ -invariant so that one can define  $\Omega$ -cohomology in the subspace (A.3).

**Lemma A.1.** *Let weights  $s_i$  be such that  $s_1 \geq s_2 \geq \dots \geq s_{n-1}$  (i.e. respective  $sl(n-1)$ -weight are nonnegative) then  $\Omega$ -cohomology vanishes in nonzero degree.*

Note that the cohomology at vanishing degree is clearly a subspace of vectors satisfying the highest-weight condition, i.e. vectors annihilated by all  $N_i^j$  with  $i < j$ .

*Proof.* The statement can be proved by induction. The first nontrivial case is  $n = 2$  where the statement immediately follows from the structure of irreducible  $sl(2)$ -modules. Suppose it is true for  $n = k$ . If  $\Omega_k$  is the respective BRST operator then  $\Omega_{k+1}$  takes the form

$$\Omega_{k+1} = \Omega_k + c^l N_l^{k+1} - \gamma_i^j (c^i \frac{\partial}{\partial c^j}), \quad (\text{A.4})$$

where there is no summation over  $k$  and summation over  $i, j, l$  runs from 1 to  $k$ . We also introduced notations  $c^i$  for  $\gamma_{k+1}^i$ . It can be rewritten as  $\Omega_{k+1} = \hat{\Omega}_k + c^l N_l^{k+1}$ , where  $\hat{\Omega}_k$  is obtained from  $\Omega_k$  by replacing  $N_i^j$  with  $\mathcal{N}_i^j = N_i^j - c^j \frac{\partial}{\partial c^i}$ . New generators form the same algebra so that  $\hat{\Omega}_k$  is also a BRST operator of the same upper-triangular subalgebra but with coefficients in a different finite-dimensional representation. In particular,  $\hat{\Omega}_k$  is nilpotent. Moreover, the induction assumption is satisfied for  $\hat{\Omega}_k$  acting in this representation. Note that the weight conditions take the form  $(\mathcal{N}_i - c^i \frac{\partial}{\partial c^i} - s_i)\Psi = 0$  in this case, where, again, no summation over  $i$  is assumed.

$\Omega_{k+1}$ -cohomology can be computed as follows. Taking as a degree minus homogeneity in  $\gamma$  one finds that  $\hat{\Omega}_k$  is the lowest (degree  $-1$ ) term in  $\Omega_{k+1}$ . The cohomological problem can be reduced to its cohomology. By the induction assumption cohomology of  $\hat{\Omega}_k$  is concentrated in degree zero (are given by  $\gamma$ -independent elements annihilated by  $\mathcal{N}_i^j$  with  $i < j$ ). The cohomology problem reduces then to the cohomology of  $c^l N_l^{k+1}$  in this subspace. But this problem is identical to that considered in [31]. Using this result and taking into account the weight condition  $s_{k+1} \leq s_i$  one concludes that the cohomology is given by  $c^i$ -independent elements annihilated by  $N_l^{k+1}$  so that the statement remains true at the next step of the induction.  $\square$

The above statement underlies the equivalence of the parent formulation based on  $\bar{\Omega}^{\text{tot}}$  and the formulation based on BRST operator  $Q_p$  and the constraints (4.5)-(4.9). Indeed, the term  $c_{IJ}T^{IJ}$  in  $\bar{\Omega}^{\text{tot}}$  implements tracelessness conditions. Reducing to its cohomology simply amounts to eliminating ghosts  $c_{IJ}$  and assuming all elements totally traceless (see [31] for details and proofs). Using then a suitable degree one can assume that the term  $\nu_i \bar{S}^i + \mu(h + \text{ghosts})$  has the lowest degree. Its cohomology can be identified with  $\mu, \nu, u_i, z$ -independent elements (see Section 5.3.1 and Lemma B.1 for notation and further details). In terms of this identification constraints  $N_i^j$  act as  $n_i^j = a_i^a \frac{\partial}{\partial a_j^a}$ , while  $Q_p$  acts as  $q_p = s_\alpha^\dagger \frac{\partial}{\partial b_\alpha} = a_\alpha^a \frac{\partial}{\partial y^a} \frac{\partial}{\partial b_\alpha}$ . The above steps are identical to those in [24] to which we refer for further details.

As a next step one takes as a degree  $\deg \gamma^i_j = -1$  so that the term implementing  $n_i^j$  has the lowest degree and we reduce the formulation to its cohomology. It follows from Lemma A.1 that the cohomology can be taken  $\gamma$ -independent. In this way one reduces the formulation to that based on  $q_p$ . At the same time, starting with the formulation based on  $Q_p$  and following [24] (or equivalently, specializing the reduction described in 5.3.1) one arrives at the same formulation by explicitly solving  $\bar{S}^i, h$  constraints.

## B AdS Weyl module: technical details

In this Appendix we collect various technical details needed for the discussion of AdS Weyl module in Section 5.3.

**Lemma B.1.** *The space of all totally traceless elements  $\Psi = \Psi(Y, A)$  satisfying*

$$(\bar{S}^p)^t \Psi = 0, \quad \bar{S}^{\hat{\alpha}} \Psi = 0, \quad h \Psi = 0, \quad (\text{B.1})$$

where  $\hat{\alpha} = p + 1, \dots, n - 1$ , is isomorphic to the space of totally traceless elements  $\Psi = \Psi(a, y, w, z)$  satisfying

$$\left( \frac{\partial}{\partial u_p} \right)^t \Psi = 0, \quad \frac{\partial}{\partial u_{\hat{\alpha}}} \Psi = 0, \quad \frac{\partial}{\partial z} \Psi = 0. \quad (\text{B.2})$$

The proof is a straightforward generalization of that from [29, 24]. The only modification has to do with taking traces into account. To generalize the recursive proof of [29, 24] to the present case one needs to show that the cohomology of the auxiliary BRST operator  $\delta = \mu \frac{\partial}{\partial z} + \nu_{\hat{\alpha}} \frac{\partial}{\partial u_{\hat{\alpha}}} + \nu_p \left( \frac{\partial}{\partial u_p} \right)^t + C_{IJ} T^{IJ}$  is trivial at nonvanishing degree in auxiliary ghost variables  $\mu, \nu, C$ . To see this one first reduces to cohomology of  $\mu \frac{\partial}{\partial z} + \nu_{\hat{\alpha}} \frac{\partial}{\partial u_{\hat{\alpha}}} + \nu_p \left( \frac{\partial}{\partial u_p} \right)^t$  and hence eliminates ghosts  $\mu, \nu$ . Using then a degree such that  $\deg u_i = -1$  the lowest degree term of the reduced differential is simply  $C_{IJ} T_0^{IJ}$  where  $T_0^{IJ}$  is obtained from  $T^{IJ}$  by omitting terms involving  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial u_i}$ . Finally,  $T_0^{IJ}$  are usual trace operators in  $d$  dimensions and hence cohomology of  $C_{IJ} T_0^{IJ}$  is concentrated at zeroth ghost degree for ghosts  $C_{IJ}$  [31]. As ghost variables  $\mu, \nu$  have been already eliminated at the previous



step one concludes that cohomology is trivial at nonvanishing degree in auxiliary ghost variables.

The next fact we need is the explicit solution to irreducibility conditions (5.8). Namely, the space of solutions to (5.8) can be isomorphically mapped to the subspace singled out by

$$(n_i^j + \delta_i^\alpha \delta_\beta^j u_\alpha \frac{\partial}{\partial u_\beta} + \delta_i^\alpha \delta_\beta^j B_\alpha^\beta) \tilde{\psi} = 0, \quad i < j. \quad (\text{B.3})$$

This can be shown by analyzing the recurrent equations obtained by substituting level- $k$  decomposition (5.6) into (5.8). Using decomposition (5.6) and separating the term  $-u_\alpha \bar{s}^{\hat{\beta}}$  by prescribing  $u_\alpha$  to carry degree 1 one recursively shows that a space of solutions to modified Young conditions (5.8) can be mapped to the subspace of elements satisfying (B.3).

In its turn the subspace (B.3) can be isomorphically mapped to the following subspace:

$$(n_i^j + \delta_i^\alpha \delta_\beta^j B_\alpha^\beta) \hat{\psi} = 0, \quad i < j. \quad (\text{B.4})$$

To see this one again substitutes decomposition (5.6) into (B.3). Solution to the resulting inhomogeneous linear equations are parameterized by elements satisfying (B.4). It is important to stress that the  $q_p$  represented in terms of parameterization (B.4) remains intact because it commutes both with  $u_\alpha$ -variables and BRST extended Young conditions (B.4).

**Proof of Proposition 5.2.** Using the above isomorphisms reduces the problem to calculating  $q_p$ -cohomology in the subspace (B.4). The  $q_p$ -cohomology problem in the subspace (B.4) is identical to that considered in [24]. Applying then lemmas 5.3 and 5.4 from [24] gives the statement.

**Proof of Proposition 5.3.** The zero-ghost-number cohomology of the total BRST operator  $Q_p$  is defined by the following chain of equivalence relations read off from (5.6) and (5.9):

$$\psi_k^{\alpha_1 \dots \alpha_k} \sim \psi_k^{\alpha_1 \dots \alpha_k} + s_\gamma^\dagger \chi_k^{\alpha_1 \dots \alpha_k | \gamma} - \hat{h} \chi_{k-1}^{\alpha_1 \dots \alpha_{k-1} | \alpha_k}. \quad (\text{B.5})$$

To fix representatives one proceeds as follows. First of all one finds representatives of the above relations without the term containing  $\hat{h}$ . These are simply representatives of  $q_p$ -cohomology at zeroth ghost number described by Proposition 5.2. Taking  $\hat{h}$  into account amounts to subtracting particular components from  $H^0(q_p)$ . To clarify which components should be cancelled out one analyzes the following residual equivalence condition:

$$s_\gamma^\dagger \chi_k^{\alpha_1 \dots \alpha_k | \gamma} - \hat{h} \chi_{k-1}^{\alpha_1 \dots \alpha_{k-1} | \alpha_k} = 0. \quad (\text{B.6})$$

One then observes that admissible  $\chi_k^{\alpha_1 \dots \alpha_k | \gamma}$  are all described by Young diagrams with  $\#y^a \geq \#a_1^a$ , where  $\#$  denote the homogeneity degree in the respective variable. More precisely, using a technique elaborated in [24] one shows that for the  $k$ -th level the homogeneity in  $y^a$  variables is  $s_1 - k \leq \#y^a \leq s_1 - 1$ . The remaining weight and Young

conditions imposed on  $\chi_k^{\alpha_1 \dots \alpha_k | \gamma}$  are such that both sides of the equivalence relations (B.5) satisfy the same algebraic constraints.

**An example:**  $n = 2$ ,  $p = 1$ ,  $t \geq 1$ . In what follows we explicitly demonstrate the  $Q_p$ -cohomology calculation for the simplest case of totally-symmetric partially-massless fields of spin  $s$  and depth  $t$  [5, 39, 6, 11].

Decomposition (5.6) takes the following form

$$\psi(a, y, w|b) = \sum_{k=0}^{t-1} \psi_k(a, y|b) u^k . \quad (\text{B.7})$$

BRST operator  $\tilde{Q}$  is given by

$$\tilde{Q}_p = s^\dagger \frac{\partial}{\partial b} - u \hat{h} \frac{\partial}{\partial b} \equiv q_p - u \hat{h} \frac{\partial}{\partial b} , \quad (\text{B.8})$$

where  $\hat{h} = n_y - B - s + t + 1$ . It acts in the subspace of (B.7) singled out by the weight constraints

$$(n_a + n_u + n_b - s) \psi(a, y, u|b) = 0 . \quad (\text{B.9})$$

Let us analyze first the cohomology in the minimal ghost number  $-1$ . For  $\psi = b\psi^1$  we get

$$s^\dagger \psi_k^1 - \hat{h} \psi_{k-1}^1 = 0 , \quad k = 0, \dots, t-1 . \quad (\text{B.10})$$

It follows that  $\psi_k^1$  consists of two parts: the kernel of  $s^\dagger$  and the particular solution determined by  $\psi_{k-1}^1$ . The exact formula reads as

$$\psi_k^1 = \tilde{\psi}_k^1 + \bar{s} \frac{\hat{h} \psi_{k-1}^1}{n_a - n_y} , \quad \text{where} \quad s^\dagger \tilde{\psi}_k^1 = 0 . \quad (\text{B.11})$$

It follows that for some elements of level  $k-1$  the denominator may vanish. This implies that these elements are set to zero.

Analyzing the above system of equations recursively results in a set of Lorentz components defined by constraints

$$s^\dagger \tilde{\psi}_k^1 = 0 , \quad (n_a - s + k + 1) \tilde{\psi}_k^1 = 0 , \quad (n_y - l) \tilde{\psi}_k^1 = 0 , \quad (\text{B.12})$$

for

$$k = 0, \dots, t-1 , \quad l = 0, \dots, s-t . \quad (\text{B.13})$$

In other words, they are described by diagrams with two rows, the first one is of length  $s-k-1$  and the length of the second row is not exceeding  $s-t$ . In manifestly  $o(d-1, 2)$  terms these are describes by a single two-row diagram with the lengths of rows  $s-1$  and  $s-t$  (see Section 5.1).

Representing the gauge parameter as  $\chi^1 = b \sum_{k=0}^{t-1} \chi_k^1(a, y) w^k$  we cast the cocycle condition into the following form

$$\psi_k^0 \sim \psi_k^0 + s^\dagger \chi_k^1 - \widehat{h} \chi_{k-1}^1, \quad k = 0, \dots, t-1. \quad (\text{B.14})$$

The term  $-\widehat{h} \chi_{k-1}^1$  defines Stueckelberg-like transformation with parameter  $\chi_{k-1}^1$  satisfying the gauge fixing condition  $s^\dagger \chi_{k-1}^1 - \widehat{h} \chi_{k-2}^1 = 0$ .

To identify representatives of the above equivalence relations we analyze them recursively starting from the level  $k = 0$ . The end result is the following collection of Lorentz tensors:

$$\psi_k^{a(s+l-k), b(s-k)}, \quad k = 0, \dots, t-1, \quad l \geq k, \quad (\text{B.15})$$

where (using the notation of Ref. [2]) a set of  $s$  symmetrized indices  $a$  is denoted by  $a(s)$  while different groups of symmetrized indices are separated by a comma. This describes AdS Weyl module for spin  $s$  and depth  $t$  gauge field. Note that it can be also described in manifestly  $o(d-1, 2)$  covariant notation, see [24, 47].

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